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Exotic Coherent Structures in the (2 + 1)-Dimensional Breaking-Soliton Equations

Zhang Jiefang[1,2]

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Hirota's bilinear form of the (2 + 1)-dimensional breaking-soliton equations introduced by Bogyovlenskii is deduced in a straightforward manner and used to construct wave-type solutions for the field variables. The peculiar localization behavior of the system by the generating dromion for the composite field variable *qr* is also brought out and is generalized to (1, *N*, 1) dromions.

The (2+1)-dimensional breaking-soliton equations

$$q_t = iq_{xy} - 2iq\partial_x^{-1}(qr)_y \tag{1a}$$

$$r_t = -ir_{xy} + 2ir\partial_x^{-1}(qr)_y \tag{1b}$$

were first studied in a series of papers by Bogoyovlenskii.[1,2] Similar equations were also studied by Calogero and Degasperis.[3] These equations were used to describe the (2 + 1)-dimensional interaction of Riemann wave propagation along the y axis with long-wave propagation along the x axis. The simplest "breaking-soliton" solution of Eq. (1), for example, can be put into the following form[1,2]:

$$(q(x, y, t), r(x, y, t)) = (Q(x, t, \lambda(y, t), R(x, t, \lambda(y, t))) \tag{2}$$

where for fixed value of $\lambda(y, t)$, (Q, R) gives a soliton solution for the nonlinear Schrödinger equation, while $\lambda(y, t)$ satisfies the Riemann wave equation. For any fixed initial value of $\lambda(y, t)$, $\lambda(y, t)$ ultimately "breaks" to

[1] Institute of Nonlinear Physics, Zhejiang Normal University, Jinhua 321004, China, and Research Center of Engineering Science, Zhejiang University of Technology, Hangzhou 310032, China.
[2] Mailing address: College of Foundation Science, Zhejiang University of Technology, Hangzhou 310032, Zhejiang, China; e-mail: zgd@public.hz.zj.cn.

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a multivalued function; thus (Q, R) also "breaks" to give a multivalued solution of Eqs. (1). Bogoyovlenskii presented the Lax pairs and the Hamiltonian structures for these equations, and showed that these equations can be solved via the inverse scattering method. Li[(4)] showed by using a recursion operator that Eqs. (1) possess an infinite set of symmetries; these symmetries constitute an infinite-dimensional Lie algebra. Li and Zhang[(5)] constructed infinitely many symmetries of Eqs. (1) by using the infinitesimal version of the "dressing" method and proved that these symmetries constitute an infinite-dimensional Lie algebra which contains some Abelian and Virasoro subalgebras. In this paper we solve the exact solitary wave solution of Eqs. (1).

To investigate the solution of Eqs. (1), we use the transformation

$$(qr)_y = V_x \tag{3}$$

where V is some arbitrary potential, so that Eqs. (1) are converted into a system of three coupled partial differential equations

$$q_t = iq_{xy} - 2iqV \tag{4a}$$

$$r_t = -ir_{xy} + 2irV \tag{4b}$$

$$(qr)_y = V_x \tag{4c}$$

In order to write these equations in the bilinear form we introduce new dependent variables by

$$q = \frac{g}{\Phi}, \qquad r = \frac{h}{\Phi}, \qquad V = -\frac{\partial^2}{\partial x \partial y} \log \Phi \tag{5}$$

where $g = g(x, y, t)$, $h = h(x, y, t)$, $\Phi(x, y, t)$ are differentiable functions, and then substitute into Eqs. (4) to obtain

$$(D_t - iD_xD_y)g \cdot \Phi = 0 \tag{6a}$$

$$(D_t + iD_xD_y)h \cdot \Phi = 0 \tag{6b}$$

$$D_x^2 \Phi \cdot \Phi = -2gh \tag{6c}$$

We now expand the functions g, h, and Φ in the form of power series as

$$\begin{aligned} g &= \varepsilon g^{(1)} + \varepsilon^{(3)} g^{(3)} + \dots \\ h &= \varepsilon h^{(1)} + \varepsilon^{(3)} h^{(3)} + \dots \\ \Phi &= \varepsilon^2 \Phi^{(2)} + \varepsilon^4 \Phi^{(4)} + \dots \end{aligned} \tag{7}$$

Substituting Eqs. (7) into Eqs. (6) and comparing various powers of ε, we obtain the following set of equations

$$\varepsilon\text{: } g_t - ig_{xy} = 0 \tag{8a}$$

$$h_t + ih_{xy} = 0 \tag{8b}$$

$$\varepsilon^2\text{: } \Phi^{(2)}_{xx} = -g^{(1)}\, h^{(1)} \tag{9}$$

$$\varepsilon^3\text{: } g^{(3)} - ig^{(3)}_{xy} = -(D_t - iD_xD_y)g^{(1)} \cdot \Phi^{(2)} \tag{10a}$$

$$h^{(3)} + ih^{(3)}_{xy} = -(D_t + iD_xD_y)h^{(1)} \cdot \Phi^{(2)} \tag{10b}$$

$$\varepsilon^4\text{: } 2\Phi^{(4)}_{xy} + D_x^2\Phi^{(2)} \cdot \Phi^{(2)} = -2(g^{(1)}h^{(3)} + h^{(1)}g^{(3)}) \tag{11}$$

and so on.

To generate line-soliton solutions, one has to first solve Eqs. (8a) and (8b) explicitly. Solving Eqs. (8a) and (8b), we have

$$g = \sum_{i=1}^{n} \exp(\xi_i), \qquad \xi_i = ik_i l_i t + k_i x + l_i y \tag{12a}$$

$$h = \sum_{i=1}^{n} \exp(\xi_i'), \qquad \xi_i' = -ik_i' l_i' t + k_i' x + l_i' y \tag{12b}$$

where k_i, k_i', l_i, l_i' are arbitrary real constants.

To generate a one-soliton solution, we put $n = 1$ to give

$$g = \exp(\xi_1), \qquad h = \exp(\xi_1') \tag{13}$$

Substituting Eqs. (13) into Eq. (9), we get

$$\Phi^{(2)}_{xx} = -\exp(\xi_1 + \xi_1') \tag{14}$$

Integrating this equation, we obtain the particular solution

$$\Phi^{(2)} = \exp(\xi_1 + \xi_1' + \psi), \qquad \exp(\psi) = -\frac{1}{(k_1 + k_1')^2} \tag{15}$$

Substituting Eqs. (13) and (15) into Eqs. (10) and (11), one can indeed choose $g^{(3)}$, $h^{(3)}$, and $\Phi^{(4)}$ to be zero, so that the series (7) truncates. Thus we have solutions of the type

$$q = \frac{g}{\Phi} = \frac{\exp(\xi_1)}{1 + \exp(\xi_1 + \xi_1' + \psi)} \tag{16a}$$

$$r = \frac{h}{\Phi} = \frac{\exp(\xi_1')}{1 + \exp(\xi_1 + \xi_1' + \psi)} \tag{16b}$$

The potential V is now described by a solitary-wave solution

$$V = \int_{-\infty}^{x} (qr)_y \, dx = -\frac{(l_1 + l_1')(k_1 + k_1')}{4} \operatorname{sec\,h}^2 \frac{1}{2} (\xi_1 + \xi_1' + \psi) \quad (17)$$

Further, from (16), we find the interesting fact that the composite field qr is described by

$$qr = -\frac{(k_1 + k_1')^2}{4} \operatorname{sec\,h}^2 \frac{1}{2} (\xi_1 + \xi_1' + \psi) \quad (18)$$

It is evident from Eqs. (17) and (18) that as the combined parameter $(k_1 + k_1') \to 0$, both the potential V and the composite field qr vanish. But, when only $(l_1 + l_1') \to 0$, the potential V alone vanishes, whereas the composite field qr, which denotes the physical quantity $\int_{-\infty}^{y} V_x \, dy'$, survives.

To generate a (1,0,1) dromion, we now take the ansatz

$$\Phi = 1 + a \exp \zeta_1 + K \exp \xi_1 \quad (19)$$

where a and K are arbitrary positive constants and ζ_1 takes the special form

$$\zeta_1 = k_1 x \quad (20)$$

Substituting Eq. (19) with (20) into Eq. (6c), one obtains

$$-k_1^2 (a \exp \zeta_1 + K \exp \xi_1) = gh \quad (21)$$

This equation suggests that the functions g and h take the form

$$g = a \exp \zeta_1 + K \exp \xi_1, \qquad h = -k_1^2 \quad (22)$$

Substituting Eq. (19) and (22) into Eq. (5), we get

$$q = \frac{g}{\Phi} = \frac{a \exp \zeta_1 + K \exp \xi_1}{1 + a \exp \zeta_1 + K \exp \xi_1}, \qquad r = \frac{h}{\Phi} = -\frac{k_1^2}{1 + a \exp \zeta_1 + K \exp \xi_1} \quad (23)$$

It can be seen that the field variables q and r are again bounded, but nondecaying along certain lines (line solitons). But the physical quantity qr is described by a dromion as

$$qr = -\frac{a k_1^2 \exp \zeta_1 + K k_1^2 \exp \xi_1}{(1 + a \exp \zeta_1 + K \exp \xi_1)^2} \quad (24)$$

which is exponentially localized with one bound state in the x direction and one bound state in the ξ_1 direction.

Similarly, we also obtain

$$q = \frac{g}{\Phi} = -\frac{k_1'}{1 + a\exp\zeta_1' + K\exp\xi_1'},$$
$$r = \frac{h}{\Phi} = \frac{a\exp\zeta_1' + K\exp\xi_1'}{1 + a\exp\zeta_1' + K\exp\xi_1'} \tag{25}$$

whereas the physical quantity qr is

$$qr = -\frac{ak_1'^2\exp\zeta_1' + Kk_1'^2\exp\xi_1'}{(1 + a\exp\zeta_1' + K\exp\xi_1')^2} \tag{26}$$

which is exponentially localized.

This can be easily generalized to multidromions. To construct the (1, 1, 1) dromion we now take

$$\Phi = 1 + a\exp\zeta_1 + b\exp\chi_1 + K\exp\xi_1, \qquad \chi_1 = l_1 y \tag{27}$$

Substituting this equation into Eq. (6c), we obtain

$$-k_1^2[a\exp\zeta_1 + K\exp\xi_1 + ab\exp(\chi_1 + \zeta_1) + bk\exp(\chi_1 + \xi_1)] = gh \tag{28}$$

This equation again suggests that

$$g = a\exp\zeta_1 + K\exp\xi_1, \qquad h = -k_1^2(1 + b\exp\chi_1) \tag{29}$$

Using Eqs. (27) and (29), we get the (1,1,1) dromion as

$$qr = -k_1^2\frac{(a\exp\zeta_1 + K\exp\xi_1)(1 + b\exp\chi_1)}{(1 + a\exp\zeta_1 + b\exp\chi_1 + K\exp\xi_1)^2} \tag{30}$$

This expression describes an exponentially localized solution with one bound state in the x direction and one bound state in the y direction as well as one bound state in the ξ_1 direction.

Similarly, we also obtain

$$qr = -k_1'^2\frac{(a\exp\zeta_1' + K\exp\xi_1')(1 + b\exp\chi_1')}{(1 + a\exp\zeta_1' + b\exp\chi_1' + K\exp\xi_1')^2},$$
$$\zeta_1' = k_1' x, \tag{31}$$
$$\chi_1' = l_1' y$$

The above analysis can be further generalized to the (1, 2, 1) dromion. It has the form

$$qr = -k_1^2 \frac{(a \exp \zeta_1 + K \exp \xi_1)(1 + b \exp \chi_1 + c \exp \chi_2)}{(1 + a \exp \zeta_1 + K \exp \xi_1 + b \exp \chi_1 + c \exp \chi_2)^2} \tag{32}$$

which represents one bound state in the x direction and two bound states in the y direction as well as one bound state in the ξ_1 direction. The (1, N, 1) dromion can be similarly obtained,

$$qr = -k_1^2 \frac{(a \exp \zeta_1 + K \exp \xi_1)(1 + \sum_{i=0}^{N} s_i \exp \chi_i)}{(1 + a \exp \zeta_1 + K \exp \xi_1 + \sum_{i=1}^{N} s_i \exp \chi_i)^2}, \qquad \chi_i = l_i y \tag{33}$$

or

$$qr = -k_1'^2 \frac{(a \exp \zeta_1' + K \exp \xi_1')(1 + \sum_{i}^{N} s_i \exp \chi_i')^2}{(1 + a \exp \zeta_1' + K \exp \xi_1' + \sum_{i}^{N} s_i \exp \chi_{i1}')^2}, \qquad \chi_i' = l_i' y \tag{34}$$

which represents one bound state in the x direction and N bound states in the y direction as well as one bound state in the ξ_1 (or ξ_1') direction.

Obviusly by taking $K = 0$, the (1, N) dromions can be easily generated.

A natural question arises of whether one can construct (N, 1, 1) dromions ($N > 1$) and then (N, M, L) dromions by extending the above procedure. Unfortunately, such an extension leads to inconsistencies and hence the problem of constructing (N, M, L) dromions remains open.

In summary, we have derived the bilinear form of the (2 + 1)-dimensional breaking-soliton equations introduced by Bogoyovlenskii and deduced the line-soliton solutions for composite field qr. We have also brought out the peculiar localization behavior of the breaking-soliton equations by the generating dromion for the physical quantity $qr = \int_{\infty}^{y} V_x \, dy$ (the composite field) and generalized it to the (1, N, 1) dromion.

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